

## THE PROBLEM OF THE TIME-OPTIMAL TURNING OF A MANIPULATOR\*

S.N. OSIPOV and A.M. FORMAL'SKII

Two types of manipulator that perform three-dimensional motions are considered, and the control problem in which the manipulator rotation is performed in minimum time is studied. The rate of rotation of a rigid body about an axis rises as the moment of inertia about this axis falls. Manipulator control amounts to a problem of the rotation of a system of rigid bodies about an axis. In addition to the angle of rotation, there is a further controlled coordinate, whose variation can vary the moment of inertia about the axis. Assuming that the moment of inertia can be stantaneously "frozen" (that pulse control signals are possible), the in-time-optimal control modes were found in /1, 2/, (see also Akulenko, L.D. et al., "Optimization of the control modes of manipulation robots", Preprint 218, In-t. Problem Mekhaniki Akad. Nauk SSSR, Moscow 1983). In these modes, the rotation occurs in the entire time interval with minimum moment of inertia about the axis of rotation. The rotation when there are constraints on the control (pulse control signals are not permitted) was considered in /3/. Numerical studies there led to the false conclusion that, in the optimal motion, with a finite number of control switchings, the moment of inertia is also a minimum throughout the time interval. Below, for a set of extreme configurations, a control is constructed for the two types of manipulator, which satisfies the Pontryagin maximum principle, when there are constraints on the control signals. During its rotation the manipulator section then performs oscillations about a position corresponding to minimum moment of inertia about the axis of rotation. It is shown that the motion considered in /3/, which contains a singular mode with minimum moment of inertia, is not optimal. The motion which satisfies the maximum principle is compared with it. There can be a singular mode in the optimal motion /4/ only when the number of control switchings is infinite.

1. The mathematical model of a manipulator with an extension pointer. We consider a manipulator (Fig.1) with three degrees of motion (1 is the fixed base, 2 is the shaft, 3 is the guide, rigidly clamped to the shaft, and 4 is the pointer). The shaft can move in a vertical direction and rotate about a vertical axis. The pointer can move horizontally along a guide. All the components are assumed to be absolutely rigid bodies. Since the equation for the manipulator vertical movement is separate from the remaining equations, we shall only consider the equations that describe the manipulator rotation and the pointer extension

$$\begin{aligned}(I + mr^2) \varphi'' + 2mrr' \varphi' &= M \\ mr'' - mr\varphi'^2 &= F\end{aligned}\tag{1.1}$$

Here,  $\varphi$  is the angle of shaft rotation,  $r$  is the pointer extension, i.e., the distance from the axis of rotation to the pointer centre of mass (the point O),  $m$  is the pointer mass, and  $I$  is the sum of the shaft plus guide moment of inertia about the axis of rotation and the moment of inertia of the pointer about the vertical through its centre of mass. The control signals are the moment  $M$  about the shaft axis of rotation and the force  $F$  extending the pointer. They have the constraints

$$|M(t)| \leq M_0, \quad |F(t)| \leq F_0 \quad (M_0, F_0 = \text{const})\tag{1.2}$$

At the initial instant, let the system be at rest in a given position

$$\varphi(0) = 0, \quad \varphi'(0) = 0, \quad r(0) = r_0, \quad r'(0) = 0\tag{1.3}$$

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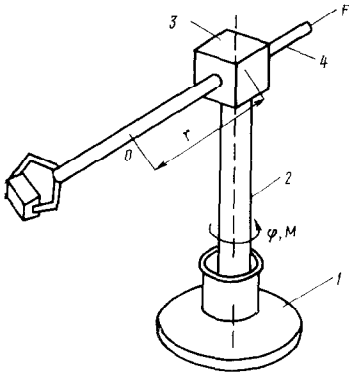


Fig.1

Our problem is to find the control functions  $M(t)$ ,  $F(t)$  such that the manipulator moves in minimum time  $T$  to a given final state

$$\begin{aligned} \varphi(T) &= \varphi_T, & \varphi'(T) &= 0, \\ r(T) &= r_T, & r'(T) &= 0 \end{aligned} \quad (1.4)$$

We change to dimensionless variables

$$\begin{aligned} r' &= r(m/I)^{1/2}, & \varphi' &= \varphi, \\ t' &= t(M_0/I)^{1/2}, \\ M' &= M/M_0, \\ F' &= (F/M_0)(I/m)^{1/2} \end{aligned} \quad (1.5)$$

Henceforth the prime is omitted.

We also use the notation

$$F_1 = (F_0/M_0)(I/m)^{1/2} \quad (1.6)$$

Using (1.5) and (1.6), relations (1.2) take the form

$$|M(t)| \leq 1, \quad |F(t)| \leq F_1$$

We change to the new variables

$$y_1 = \varphi, \quad y_2 = \varphi'(I + mr^2), \quad y_3 = r, \quad y_4 = r' \quad (1.7)$$

The primes here denote differentiation with respect to dimensionless time, and  $y_2$  is the angular momentum about the vertical axis.

In Cauchy's form the equations of motion (1.1) are

$$\begin{aligned} y_1' &= y_2/(1 + y_3^2), & y_2' &= M \\ y_3' &= y_4, & y_4' &= y_3 y_2^2/(1 + y_3^2) + F \end{aligned} \quad (1.8)$$

The boundary conditions (1.3), (1.4) take the form in variables (1.7):

$$\begin{aligned} y_1(0) &= 0, & y_2(0) &= 0, & y_3(0) &= r_0, & y_4(0) &= 0 \\ y_1(T) &= \varphi_T, & y_2(T) &= 0, & y_3(T) &= r_T, & y_4(T) &= 0 \end{aligned}$$

Consider the case when the initial and final positions of the pointer are the same, i.e.,

$$y_1(0) = y_1(T) \quad (r_0 = r_T) \quad (1.9)$$

Under conditions (1.9) the extreme configurations of the manipulator are symmetric to one another about the plane  $y_1 = \varphi_T/2$ .

Pontryagin's function  $[H]$  is

$$H = M\psi_2 + F\psi_4 + \frac{y_2\psi_1}{1 + y_3^2} + y_4\psi_3 + \frac{y_3 y_2^2 \psi_4}{(1 + y_3^2)^2}$$

We write the system of equations for the conjugate variables

$$\begin{aligned} \psi_1' &= 0, & \psi_2' &= \frac{-\psi_1}{1 + y_3^2} - \frac{2y_2 y_3 \psi_4}{(1 + y_3^2)^2} \\ \psi_3' &= \frac{2y_2 y_3 \psi_1}{(1 + y_3^2)^2} + \frac{y_2^2 \psi_4 (3y_3^2 - 1)}{(1 + y_3^2)^3}, & \psi_4' &= -\psi_3 \end{aligned} \quad (1.10)$$

The optimal control must satisfy the conditions

$$M(t) = \text{sign } \psi_2(t), \quad F(t) = F_1 \text{ sign } \psi_4(t) \quad (1.11)$$

If  $y_i(t)$ ,  $\psi_i(t)$  ( $i = 1, 2, 3, 4$ ) is the solution of system (1.8), (1.10) with certain functions  $M(t)$ ,  $F(t)$ , then  $-y_1(T-t)$ ,  $y_2(T-t)$ ,  $y_3(T-t)$ ,  $-y_4(T-t)$ ,  $\psi_1(T-t)$ ,  $-\psi_2(T-t)$ ,  $-\psi_3(T-t)$ ,  $\psi_4(T-t)$  are also the solution of this system with functions  $-M(T-t)$ ,  $F(T-t)$ .

Under conditions (1.9) we shall henceforth seek the solution of the problem that has the symmetry properties

$$\begin{aligned} y_1(t) &= \varphi_T - y_1(T-t), & y_2(t) &= y_2(T-t) \\ y_3(t) &= y_3(T-t), & y_4(t) &= -y_4(T-t) \\ M(t) &= -M(T-t), & F(t) &= F(T-t) \\ \psi_1(t) &= \psi_1(T-t), & \psi_2(t) &= -\psi_2(T-t) \\ \psi_3(t) &= -\psi_3(T-t), & \psi_4(t) &= \psi_4(T-t) \end{aligned} \quad (1.12)$$

Notice that, corresponding to the above time-optimal problem, we have the problem of maximizing the angle of rotation  $y_1(T)$  for a given time  $T$ .

2. Simplified equations of motion. Consider the case when the non-linear term describing the centrifugal force can be neglected in the last equation of (1.8). This is possible when the shaft rotates slowly and the control force  $F$  is much greater in absolute value than the centrifugal force. Then, Eqs. (1.8) take the form

$$y_1' = y_2/(1 + y_3^2), \quad y_2' = M, \quad y_3' = y_4, \quad y_4' = F \quad (2.1)$$

The equations for the conjugate variables now become

$$\begin{aligned} \psi_1' &= 0, \quad \psi_2' = -\psi_1/(1 + y_3^2) \\ \psi_3' &= 2y_2y_3\psi_1/(1 + y_3^2)^2, \quad \psi_4' = -\psi_3 \end{aligned} \quad (2.2)$$

The last two equations of (2.1) are separated, so that it is easier to study system (2.1) than the complete system (1.8).

For the simplified Eqs. (2.1), the optimal control is likewise described by (1.11). It follows from (2.1), (2.2), and (1.11) that, when  $y_2(0) = y_2(T)$ , the optimal control  $M(t)$  has just one switching, which occurs at the instant  $T/2$ . The form of control  $F(t)$  depends on the ratios between the parameters  $F_1$ ,  $\varphi_T$ , and  $r_0$ .

The moment of inertia of the system relative to the shaft axis of rotation is a minimum when the centre of mass  $O$  of the pointer is on this axis. The manipulator motion during the finite time interval when the centre of mass of the pointer remains on this axis will be called singular (Fig.2). The relevant control  $F(t) = 0$  in this interval will also be called singular. With the motion shown in Fig.2, the centre of mass of the pointer coincides with the axis of shaft rotation for a minimum time  $t_1$  ( $y_3(t_1) = y_4(t_1) = 0$ ). By Eqs. (2.1), the control  $F(t)$  which realizes this pointer displacement has one switching at the instant  $t_1/2$  (Fig.2). After this movement the shaft rotates with the pointer fixed, up to the instant  $t_2$  (singular motion). With  $t_2 \leq t \leq \tau$ , the motion is symmetric to the motion when  $0 \leq t \leq t_1$  (see (1.12)). The control  $M(t)$  has one switching at the instant  $\tau/2$ .

The necessary and sufficient condition for the singular mode to be possible is that the ratio between the parameters  $F_1$  and  $r_0$  should be such that the pointer centre of mass can reach the axis of shaft rotation at zero velocity during the manipulator rotation through the angle  $\varphi_T$ . This ratio is found by integrating system (2.1) and is given by

$$\begin{aligned} 2\alpha < \varphi_T F_1, \quad \alpha = \arctg r_0 - 2 \arctg^{1/2} r_0 + \sqrt{r_0} \times \\ \left[ \frac{1}{2} \ln \frac{1 + (1 + \sqrt{r_0})^2}{1 + (1 - \sqrt{r_0})^2} + \arctg(\sqrt{r_0} - 1) + \arctg(\sqrt{r_0} + 1) \right] \end{aligned} \quad (2.3)$$

The time of motion shown in Fig.2 is given by

$$\tau = 2 [\varphi_T + 4r_0/F_1 - 2x/F_1]^{1/2}$$

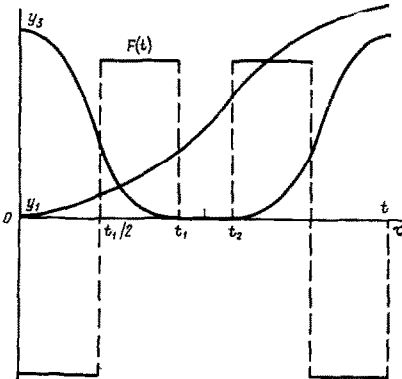


Fig.2

It is claimed incorrectly in [3], for the system of Eqs. (1.8), that a motion similar to that shown in Fig.2 is time-optimal. We shall first show that this motion does not satisfy the maximum principle for the simplified system (2.1), and hence it is not optimal for it. (We show in Sect.3 that the motion is likewise not optimal for the initial system (1.8)).

Let the motion satisfy the maximum principle. The control  $M(t)$  has one switching, so that, if it satisfies the first of conditions (1.11), then  $\psi_1 = \text{const} > 0$ ,  $y_2(t) > 0$  for  $0 \leq t \leq \tau$ . In this motion we have  $y_3 = 0$  for  $t_1 \leq t \leq t_2$  and  $y_4 > 0$  for  $t \in [t_1, t_2]$ . Thus  $\psi_3'(t) \equiv 0$  for  $t_1 \leq t \leq t_2$  and  $\psi_3'(t) > 0$  for  $t \in [t_1, t_2]$ . The solutions of Eqs. (2.2) have the symmetry property (1.12), since the motion has this property. Then, for  $t_1 \leq t \leq t_2$  the function  $\psi_3(t) \equiv 0$ , while for  $0 \leq t < t_1$  the function  $\psi_3(t)$  is strictly monotonically increasing and negative, and for  $t_2 < t \leq \tau$  it is strictly monotonically increasing and positive.

By the last equation of system (2.2), the function  $\psi_4(t)$ , which is symmetrical about the line  $t = \tau/2$ , either nowhere vanishes, or else vanishes only when  $t_1 \leq t \leq t_2$  (identically), or has just two isolated zeros. In none of these conceivable cases does the control  $F(t)$  shown in Fig.2 satisfy the second of conditions (1.11), i.e., the maximum principle.

For certain values of the parameter  $F_1 \in [0.5; 12]$ ,  $\varphi_T = 2$ , and a number of values of  $r_0$ , we can construct numerically the control  $F(t)$ ,  $M(t)$ , and the corresponding motion which satisfies the maximum principle. The algorithm for constructing the motion is as follows. We will first construct the admissible control  $F(t)$ , symmetrical about the line  $t = T/2$ , with three intervals of constancy, and the  $M(t)$  (Figs.3, 4) which takes the manipulator from

the initial to the final position. Then, using the conditions  $\psi_2(T/2) = 0, \psi_3(T/2) = 0, \psi_4(t_p) = 0$ , where  $t_p$  is the instant of first switching the control, we find numerically the initial values of the conjugate variables. By solving systems (2.1) and (2.2) simultaneously, we show that the control in question satisfies the maximum principle.

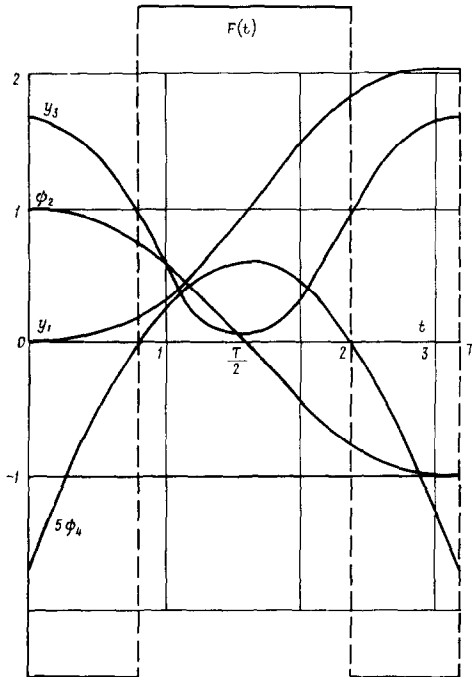


Fig. 3

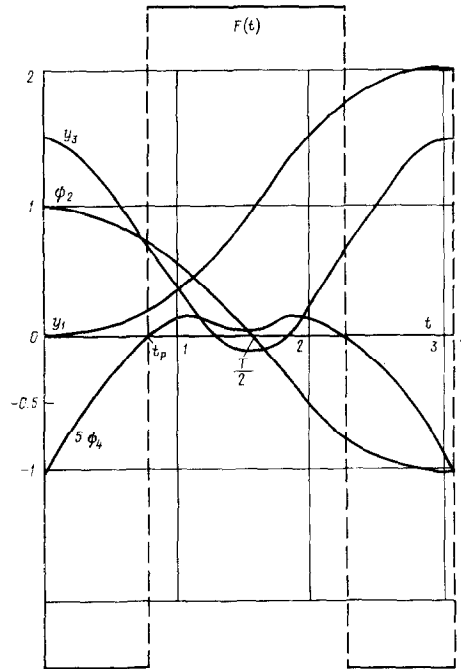


Fig. 4

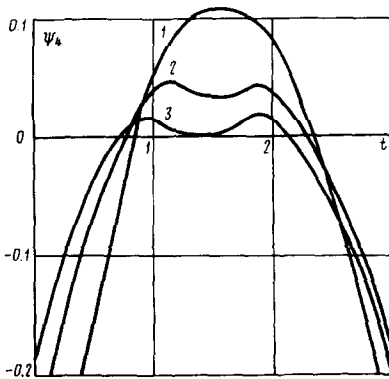


Fig. 5

Let us describe the results of a numerical study when  $F_1 = 2.5$ .

For  $r_0 > 1.553$ , condition (2.3) does not hold, and the motion shown in Fig. 2 is impossible. The motion which satisfies the maximum principle for these values of  $r_0$ , together with the control  $F(t)$  and the conjugate variables  $\psi_2(t), \psi_4(t)$ , are shown in Fig. 3 ( $r_0 = 1.63$ ). In this motion, the centre of mass of the pointer does not reach the axis of shaft revolution.

For  $r_0 \approx 1.553$ , relation (2.3) becomes an equation. In the motion which satisfies the maximum principle,  $r(T/2) = 0$ , while the derivatives  $\psi_4'(T/2) = \psi_4''(T/2) = \psi_4'''(T/2) = 0$ .

On further reducing  $r_0$  oscillations of the pointer centre of mass about the vertical occur in the motion. A small depression appears in the graph of  $\psi_4(t)$  at  $t = T/2$ . For  $r_0 = 1.41$ , the motion and the control  $F(t)$  are shown in Fig. 4; the time  $T = 3.094$ ; a motion that includes a singular piece occurs during time  $\tau = 3.096$ .

The first switching of the control  $F(t)$  shown in Fig. 4 occurs later than the first switching of the control with a singular piece ( $t_p > t_1/2$ ) which is shown in Fig. 2. On further reducing  $r_0$ , the depression in the graph of  $\psi_4(t)$  increases, and for  $r_0 \approx 1.389$  the bottom of the depression touches the abscissa axis ( $\psi_4(T/2) = 0$ ). The change in the nature of  $\psi_4(t)$  as  $r_0$  decreases can be seen from Fig. 5, where we plot curves with  $r_0 = 1.63; 1.47$ ; and  $1.389$ . The behaviour of  $\psi_4(t)$  suggests that, for  $r_0 < 1.389$ , extra switchings of the control  $F(t)$  must appear (no numerical studies were made with  $r_0 < 1.389$ ).

Some of our numerical results are shown in Table 1. The first four rows contain the results for the simplified system, and the last two, the results for the complete system, described in Sect. 3. In the table,  $r_{01}$  is the upper bound of the values of  $r_0$  at which a singular mode is possible,  $r_{02}$  is the lower bound of the  $r_0$  for which the optimal control contains three intervals of constancy,  $r_0'$  is the value of  $r_0$  at which the difference  $\tau - T$  for the given  $F_1$  is a maximum, and  $\tau$  and  $T$  are the time of singular motion and the optimal time with  $r_0 = r_0'$ . It is clear from the table that, as  $F_1$  increases,  $\tau - T$  increases.

For  $F_1 = 11.7$ , it amounts to roughly 1% of the time  $\tau$ .

$F_1$	$r_{01}$	$r_{02}$	$r_{03}$	$T$	$(\tau - T) \cdot 10^3$
1	0.521	0.482	0.487	2.8778	0.6
2.5	1.553	1.389	1.41	3.091	5
6.7	6.85	5.77	5.85	3.807	28
11.7	17.49	14.56	14.69	4.529	43
1	0.525	0.478	0.488	2.87880	0.58
2.5	1.61	1.388	1.45	3.10744	3.5

For the simplified system of equations, the corresponding problem of maximizing the angle of rotation with a fixed  $T$  amounts to maximizing the integral

$$\int_0^T \frac{f(\xi) d\xi}{1 + y_3^2(\xi)}, \quad f(t) = \begin{cases} t, & 0 \leq t \leq T/2 \\ T - t, & T/2 \leq t \leq T \end{cases}$$

where  $y_3(t)$  is the solution of the equation  $y_3'' = F$  with  $|F(t)| \leq F_1$  and the boundary conditions  $y_3(0) = y_3(T) = r_0$ ,  $y_3'(0) = y_3'(T) = 0$ .

This recalls Fuller's problem /6, 7/, in which, for sufficiently small  $r_0$ , the optimal motion contains a singular mode which is bordered on both sides by chattering modes. In each of the latter the control has an infinite number of switching instants which accumulate to a condensation point. It was shown in /4/ that a similar situation occurs in the time-optimal problem for sufficiently small values of  $r_0$ .

3. The complete equations. We consider the initial system of equations (1.8) and a motion similar to that shown in Fig.2. The control  $M(t)$  then switches once at the instant  $\tau/2$ . The pointer centre of mass then comes into coincidence with the shaft axis of rotation in minimum time  $t_1$ . This motion may not be time-optimal (though the contrary is claimed in /3/), since the necessary condition for optimality of the conjugation of the singular and non-singular pieces of the control  $F(t)$  at its points of discontinuity  $t_1, t_2$  is not satisfied (/8, p.239; /9, p.234; /10, 11/). This condition may be expressed as follows. For the conjugation to be optimal, its first negative value at the point of conjugation of the singular and non-singular controls must be taken by the quantity

$$L = (-1)^k \frac{\partial}{\partial F} \left( \frac{d^{2k}}{dt^{2k}} \frac{\partial H}{\partial F} \right)$$

for an odd value of the index  $k$ . Analytic evaluation of the derivatives shows that, in the interval  $[t_1, t_2]$ ,  $L = 0$  for  $k = 1$  and  $L = -2y_2\psi_1$  for  $k = 2$ . Since  $y_2(t) > 0$  and  $\psi_1 = \text{const} > 0$  for  $0 \leq t \leq \tau$ , then  $L < 0$  for  $k = 2$ .

Note that this necessary condition for optimality does not hold for the simplified system (2.1), (2.2). If  $M(t)$  has one switching, then the control  $F(t)$ , which has any finite number of switchings and contains a singular piece, likewise may not be optimal.

As a result of numerical studies of system (1.8), carried out for the values  $F_1 = 1$ , and 2.5, and  $\varphi_T = 2$ , we obtained the control and the corresponding motion that satisfies the maximum principle. If  $F_1 = 2.5$  and  $r_0 > 1.61$ , then the control  $F(t)$  is switched twice, and the pointer centre of mass does not reach the axis of rotation during the motion. Similar motions were obtained in /3/. For  $r_0 \approx 1.61$ , the pointer centre of mass touches the axis of shaft rotation during the motion. For  $1.388 < r_0 < 1.61$ , the control  $F(t)$  again switches twice, and the centre of mass performs oscillations about the axis of shaft rotation, though it does not stay on the axis for a finite time, as was claimed in /3/. The functions  $y_i(t)$ ,  $\psi_i(t)$  ( $i = 1, 2, 3, 4$ ),  $M(t)$ , and  $F(t)$  behave qualitatively in the same way as for the system of Eqs.(2.1) and (2.2) (Figs.3-5).

In the same way as for the simplified equations, the variation of  $\psi_1(t)$  as  $r_0$  decreases suggests the appearance of new switchings of the control  $F(t)$  (Fig.5).

Some of our numerical results are shown in the last two rows of Table 1. It can be seen that calculations for the simplified and the complete systems give closely similar results. With the chosen manipulator parameters, the relative time advantage of the motion with oscillations over the motion with a singular piece is small.

It seems that we can explain as follows the fact that, in oscillatory motion of the pointer (Fig.4), the manipulator movement is faster than in the case of motion with a singular piece (Fig.2). When the pointer centre of mass is in coincidence with the axis of rotation for a minimum time, the pointer initially accelerates and then brakes. During the acceleration the moment of inertia falls at a maximum rate, while during the braking the moment of inertia falls more slowly. In the case of motion with oscillations, the moment of inertia likewise falls initially at a maximum rate, but the braking occurs later than in the case of departure to a singular mode. In the case of motion with oscillations, the moment of inertia is, so to

speak, less "on average" than with motion with a singular piece, and the former motion is preferable to the latter in the sense of time-optimality.

4. A manipulator with a rotating section. Consider (Fig.6) a manipulator that consists of a moving base 1, hinged to a fixed bed 2, and a Sect.3. The base and section are connected by a cylindrical hinge whose axis is horizontal. The manipulator has two degrees of freedom, corresponding to rotation of the base about a vertical axis and rotation of the section about the horizontal. As the generalized coordinates we take the angle of base rotation  $\varphi$ , and the angle of section rotation  $\varphi_1$ , measured from the vertical (Fig.6). The system control is by the moments of the forces  $M$  and  $M^1$ , applied to the axes of base and section rotation.

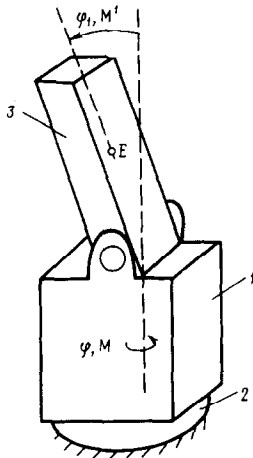


Fig.6

The equations of motion of the system are (see Osipov S.N., "The problem of control of manipulator rotation", Dep. at VINITI 31.12.86, No.489-B 87, Moscow, 1987)

$$\begin{aligned} [J + B + (m\rho^2 + C - B) \sin^2 \varphi_1] \varphi'' + (m\rho^2 + C - B) \sin 2\varphi_1 \dot{\varphi} \dot{\varphi}_1 &= M \\ (m\rho^2 + A) \varphi_1'' - \frac{1}{2} (m\rho^2 + C - B) \sin 2\varphi_1 \dot{\varphi}^2 &= M^1 + mg\rho \sin \varphi_1 \end{aligned}$$

Here,  $J$  is the base moment of inertia about the vertical,  $m$  is the section mass,  $\rho$  is the distance from the axis of section rotation to its centre of mass  $E$ , while  $A$ ,  $B$ , and  $C$  are the moments of inertia of the section in the principal central axes of inertia,  $A$  is the moment of inertia about a line parallel to the axis of section rotation and through the point  $E$ ,  $B$  is the moment of inertia about the longitudinal axis of the section, and  $g$  is the acceleration due to gravity.

We impose on the control moments the bounds

$$|M(t)| \leq M_0, \quad |M^1(t)| \leq M_0^1$$

At the initial instant the system is in the configuration

$$\varphi(0) = 0, \quad \dot{\varphi}(0) = 0, \quad \varphi_1(0) = \varphi_{10}, \quad \dot{\varphi}_1(0) = 0$$

We wish to find the control functions such that the system passes in minimum time  $T$  into the final configuration

$$\varphi(T) = \varphi_T, \quad \dot{\varphi}(T) = 0, \quad \varphi_1(T) = \varphi_{1T}, \quad \dot{\varphi}_1(T) = 0$$

Additionally, we take  $\varphi_1(0) = \varphi_1(T)$ . This condition is similar to (1.9), and under it we seek the solution that satisfies the symmetry conditions (1.12).

Studies similar to the above for a manipulator with extension pointer were made for a manipulator with rotating section. A change of variables similar to (1.5), (1.7) was made. We studied the simplified equations obtained after neglecting the moment of the centrifugal force. The singular motion for this system is rotation of the base with a vertical disposition of the section. It was shown analytically that the motion with one switching of the control moment  $M(t)$  and a finite number of switchings of the moment  $M^1(t)$  containing a singular piece, may not be optimal either for the simplified or for the complete system of equations. For both these systems we constructed numerically, for certain values of the dimensionless parameters, the control and motion that satisfy the Pontryagin maximum principle. The section then performs oscillations about the vertical, while the moment  $M(t)$  switches once.

Comparison of the results for the two types of manipulator shows that their optimal motions are of the same type.

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## MODES WITH SWITCHINGS OF INCREASING FREQUENCY IN THE PROBLEM OF CONTROLLING A ROBOT\*

V.F. BORISOV and M.I. ZELIKIN

Trajectories that are optimal with respect to high-speed response are constructed for a system for controlling a two-component manipulator (a robot). It is shown that when the initial conditions lie within a certain open region of the phase space, all optimal trajectories will have a segment of switchings of increasing frequency (SIF), i.e. a segment in which the control will undergo an infinite number of switchings in a finite time interval.

The synthesis of the optimal control in the  $R^2$  plane containing the mode of SIF was first constructed by Fuller /1/. It was shown in /2/ that the synthesis is structurally stable in the sense that adding terms of higher order of smallness to the integrand and to the right-hand sides of the system of differential constraints does not affect the qualitative pattern of the optimal synthesis in the neighbourhood of the origin of coordinates.

The present paper explains that the synthesis in the problem of optimal control (relative to the high speed response) of the motion of the robot appears, in a certain sense, a direct product of the synthesis appearing in the Fuller problem and of the synthesis in the simplest problem of high-speed response (/3/, pp.38-47). The special aspect of the present paper consists of the proof of the proposition that switching surface is a piecewise-smooth manifold. The presence of the SIF mode is connected only with the fact that every trajectory intersects this surface an infinite number of times. In existing papers, the piecewise smoothness of the switching curve was proved for the two-dimensional problems using the SIF mode only for problems admitting of a one-parameter group of symmetries /1, 4-6/. A proof of the presence of SIF was given in /7, 8/.

1. Formulation of the problem. The problem of controlling the robot can be formulated in two ways /9/. A movable element is fixed on a massive vertical cylinder rotating about its axis. In the first version the movable element has the form of a bar rotating in the vertical plane, and in the second version it takes the form of a horizontal advancing arrow. The system has two control parameters, the moment acting on the vertical cylinder, and

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